ARTICLES

Selection of a desirable chaotic phase using small feedback control

Yoshihiko Nagai
Department of Physics and Astronomy, The University of Kansas, Lawrence, Kansas 66045

Ying-Cheng Lai
Department of Physics and Astronomy and Department of Mathematics, Kansas Institute for Theoretical and Computational Science, The University of Kansas, Lawrence, Kansas 66045

(Received 8 December 1994; revised manuscript received 13 February 1995)

It is common for nonlinear dynamical systems to exhibit behaviors where orbits switch between distinct chaotic phases in an intermittent fashion. A feedback control strategy using small parameter perturbations is proposed to stabilize the trajectory around a desired chaotic phase. The idea is illustrated by using an intermittent chaotic time series generated by model dynamical systems in parameter regimes after critical events such as the interior crisis. Relevance to biological situations is pointed out.

PACS number(s): 05.45.+b

I. INTRODUCTION

In this paper, we consider the following situation. Suppose there is a nonlinear dynamical system whose orbits switch intermittently between distinct chaotic phases. Suppose further that one of these chaotic phases corresponds to a desired operational state of the system. The question we address is, can one apply small feedback control to one of the available system parameters to keep trajectories originating from random initial conditions in the desired chaotic phase so as to avoid the other chaotic phases which may correspond to undesired operational states of the system?

The intermittent chaotic behavior described above arises commonly in nonlinear dynamical systems. For example, a dynamical system in a parameter regime after a bifurcation called the "interior crisis" [1–3] exhibits such intermittent chaotic behavior. The phenomenology of the interior crisis is as follows. Before the crisis, there is a chaotic attractor and a coexisting nonattracting chaotic saddle in the phase space. The chaotic attractor and the chaotic saddle are separated from each other and, hence, trajectories originating from almost all initial conditions eventually asymptote to the chaotic attractor. At the crisis, the chaotic saddle collides with the chaotic attractor so that the original nonattracting chaotic saddle becomes part of the combined attractor, whose phase-space extent is larger than the original chaotic attractor. After the crisis, trajectories wander on the whole combined larger attractor in such a way that the trajectories visit both parts, which correspond to the original chaotic attractor and the chaotic saddle, in an intermittent fashion. As a consequence, the time series recorded from such a trajectory exhibits distinct intermittent chaotic phases. As an example, Fig. 1 shows a time series recorded from the Ikeda map [4] at a parameter value after an interior crisis, where x_n versus n is plotted; x_n is a state variable and n is the discrete time (details will be described in Sec. III). The intermittent chaotic behavior in Fig. 1 is clear, where there are two distinct chaotic phases, one being the chaotic signal confined approximately within $-0.1 \le x_n \le 1.1$ and the other the chaotic signal outside this range of x_n values. Our goal is to devise a scheme to keep trajectories in one of the chaotic phases by applying only small parameter perturbations to the system.

Our work is motivated by the fact that intermittent chaotic signals also arise in biomedical systems. For instance, the electroencephalogram (EEG) or the electrocorticogram (ECoG) signals measured from patients with certain types of epileptic seizures exhibit distinct random phases [5]. Under normal physiological condition, the EEG or ECoG signal appears to be random. When the

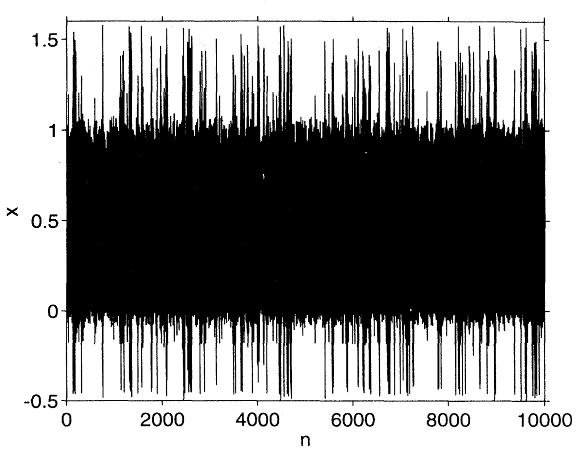

FIG. 1. An intermittent chaotic time series generated by the Ikeda-Hammel-Jones-Moloney map.

1063-651X/95/51(5)/3842(7)/$06.00

seizure strikes, the EEG or ECoG signal appears to exhibit a distinct, qualitatively different random behavior with signal amplitude much larger than that of the random signal under normal condition. Evidence now indicates that deterministic chaos may play a role in these random EEG and ECoG signals [6]. Thus, signals in these random phases are likely to be deterministically chaotic. The small amplitude chaotic signal would correspond to a "healthy" state [7], while the large amplitude chaotic signal may correspond to the seizure state, or "sick" state. It is then desirable to keep the signal in the healthy chaotic phase to prevent seizures from occurring by applying small perturbations. While this is a rather ambitious goal, our idea may provide alternative insight into the study of therapeutical techniques for diseases such as the epileptic seizures [8]. We emphasize, however, that the claim of "chaos in EEG or ECoG" (or "chaos in the brain" [9]) has not yet been established with the current available data, although preliminary evidence based on the time series analysis suggests some signature of the chaotic behavior in the brain [8,9]. Such evidence usually consists of calculations of the correlation dimension or the Lyapunov exponents from the EEG and ECoG signals. Whether or not these calculations are reliable depends on the validity of the hypothesis that the EEG and ECoG signals are generated by stationary deterministic chaotic processes. Such an hypothesis remains questionable because of the nonstationary characteristics apparently present in typical EEG and ECoG signals [10]. The work reported in this paper deals with low-dimensional deterministic chaotic systems. Therefore, at present it is not clear whether our work can be applied to more complicated random time series such as the EEG and ECoG signals.

Control of chaos by stabilizing unstable periodic orbits embedded in a chaotic attractor has been proposed by Ott, Grebogi, and Yorke (OGY) in 1990 [11]. The basic idea of the OGY method is as follows. First, one chooses an unstable periodic orbit embedded in the attractor, the one that yields the best system performance according to some criterion. Second, one defines a small region around the desired periodic orbit. For chaotic attractors, a trajectory originating from a random initial condition will come arbitrarily close to the unstable periodic orbit at some later time. This time scales with the size of the small neighborhood around the periodic orbit as a power law [11]. When this occurs, small judiciously chosen temporal parameter perturbations are applied to force the trajectory to approach the unstable periodic orbit because, without control, the trajectory will subsequently leave the periodic orbit. This method is extremely flexible because it allows for the stabilization of different periodic orbits, depending on one's needs, for the same set of nominal values of the parameter. This idea has since stimulated many theoretical investigations [12] and has been successfully applied to various physical [13], chemical [14], and biological [15] systems.

Our method of confining trajectories in the desired chaotic phase is based on OGY's idea of controlling chaos. The strategy is to construct a long target chaotic orbit that lives only on the part of the attractor corresponding to the desired chaotic phase. By stabilizing trajectories around the target orbit, the part of the attractor that corresponds to the undesired chaotic phases can be avoided. This can indeed be achieved if the target chaotic orbit is an approximately continuous trajectory generated by the evolution equations of the dynamical system. Such a target orbit possesses a complete geometric structure of stable and unstable directions that exist for typical chaotic trajectories. Small, time-dependent parameter perturbations can then be applied to keep trajectories originating from random initial conditions in the neighborhood of the target orbit. The construction of such a target orbit is, therefore, a crucial step in our controlling method. This will be detailed in Sec. II. We mention that a method for stabilizing chaotic orbits on the attractor has been proposed and applied to synchronization of chaotic systems [16], and a method for generating desired chaotic orbits has been proposed for one-dimensional maps [17].

This paper is organized as follows. In Sec. II, we describe our method of constructing a target orbit in the desired chaotic phase. We also discuss the application of the OGY idea to stabilize trajectories around the target orbit. Numerical results using the Ikeda-Hammel-Jones-Moloney map [4] in a parameter regime after an interior crisis are presented in Sec. III. In Sec. IV, we present conclusions and general discussions.

II. CONSTRUCTION OF THE TARGET CHAOTIC ORBIT AND THE CONTROL METHOD

Our goal is to construct an arbitrarily long target orbit that resides only on some nontrivial subset of the chaotic attractor. We consider the situation where the dynamical system possesses a chaotic attractor with a natural invariant measure. Thus, trajectories on the attractor are ergodic and, almost every initial condition in the basin of attraction of the attractor generates orbits that wander on the whole attractor. It can be rigorously proven that there exist ergodic invariant measures (i.e., the natural measures) for one-dimensional maps under fairly general conditions, and it is believed that such ergodic measures also exist for chaotic attractors of higher-dimensional systems [18]. If we evolve the system without any intervention (i.e., either small phase space or parameter space perturbations), the chaotic trajectory will come arbitrarily close to any points on the whole attractor. While it is possible to find short orbits, namely, orbits whose lengths are less than the average time a typical chaotic trajectory spends in the desirable part of the attractor (see Fig. 1), it is not possible in practice to find arbitrarily long orbit wandering only on the desirable part of the attractor. One may choose different initial conditions to search for such a long target orbit, but the search time for these orbits, if they exist, will be prohibitively long. Therefore, it is necessary to interrupt the chaotic trajectory in a time-dependent fashion to obtain an arbitrarily long target orbit. The strategy we will use in this paper is similar to the technique of "targeting" [19] which directs a chaotic orbit originating from a given initial condition rapidly to another given point on the chaotic attractor. In our case, the idea is to monitor the trajectory so that whenever it starts to escape the desirable part of the attractor, say, at time t_2, we go back to some earlier time t_1, where $t_1 < t_2$, to apply small perturbations to the orbit so that the new orbit starting from t_1 still stays on the desirable part of the attractor within time t_2. This can indeed be achieved due to the "butterfly" effect of the chaotic attractor, i.e., sensitive dependence of chaotic trajectories on small perturbations.

Our method is detailed as follows. Given a chaotic system, trajectories starting from almost all initial conditions eventually visit all part of the chaotic attractor. Suppose that at time t we give a small perturbation ϵ to the trajectory. At time $t+T$, where $T \sim \ln(L/\epsilon)/\lambda_{\max}$ (L is the size of the chaotic attractor and $\lambda_{\max}$ is the largest Lyapunov exponent), the perturbation grows to the whole extent of the attractor and, consequently, the trajectory is completely altered by this time. Our algorithm for constructing a target orbit consists of several steps. First, we choose a random initial condition and iterate it for a certain length of the time to get rid of the transient and to land the trajectory on the desired part of the chaotic attractor. Denote the trajectory point after the initial transient to be $\mathbf{y}_0$. Next, we iterate $\mathbf{y}_0$ for Δt (the observation time) and monitor the trajectory to see if it falls on the undesired part of the attractor in Δt. If not, $\mathbf{y}_0$ is taken as the first point on the target orbit. If it does, a small random perturbation ϵ is applied to $\mathbf{y}_0$ to yield a new starting trajectory point $\mathbf{y}'_0$. This process is repeated until all trajectory points starting from $\mathbf{y}_0$ within time Δt fall on the desired part of the attractor. The perturbed starting point $\mathbf{y}'_0$ is then taken as the first point $\mathbf{y}_0$ on the target orbit. For the next point on the target orbit, we iterate $\mathbf{y}_0$ once to get $\mathbf{y}_1$ and perform the monitoring process for $\mathbf{y}_1$ until trajectory points starting from $\mathbf{y}_1$ all fall on the desired part of the attractor within the observation time Δt. The entire process is repeated and, in principle, arbitrarily long target chaotic orbits can be obtained. Note that the observation time Δt should be at least T, the time required for the small perturbation ϵ to grow to the size of the attractor.

After an appropriate target orbit is obtained, we can apply the OGY idea of controlling chaos to stabilize it. Consider a chaotic system that can be described by two-dimensional maps on the Poincaré surface of section,

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n, p) , \quad (1)$$

where $\mathbf{x}_n \in \mathbb{R}^2$, p is an externally controllable parameter. For p values considered in this paper, we assume that trajectories starting from almost all initial conditions exhibit intermittent chaotic behavior and there are distinct parts of the chaotic attractor corresponding to desired and undesired phases. In the spirit of the OGY idea, we require that the parameter perturbations be small, i.e.,

$$|\Delta p| \equiv |p - p_0| < \delta , \quad (2)$$

where p_0 is some nominal parameter value and δ is a small number defining the range of parameter perturbations. Let $\{\mathbf{y}_n\}$ ($n = 0, 1, 2, \ldots, N$) be a long target orbit on the desired part of the chaotic attractor. Now generate the trajectory $\{\mathbf{x}_n\}$ to be stabilized around the target orbit. Randomly pick an initial condition $\mathbf{x}_0$ and assume that the trajectory point $\mathbf{x}_n$ ($n \geq 0$) falls in a small neighborhood of the point $\mathbf{y}_k$ of the target orbit at time step n. Without loss of generality, we set $k = n$ on the target orbit. In this small neighborhood, the linearization of Eq. (1) is applicable. We have, thus,

$$\mathbf{x}_{n+1}(p_n) - \mathbf{y}_{n+1}(p_0) = \mathbf{J}\cdot[\mathbf{x}_n(p_0) - \mathbf{y}_n(p_0)] + \mathbf{K}\Delta p_n , \quad (3)$$

where $\Delta p_n = p_n - p_0$, $|\Delta p_n| \leq \delta$, $\mathbf{J}$ is the 2×2 Jacobian matrix and $\mathbf{K}$ is a two-dimensional column vector,

$$\mathbf{J} = \mathbf{D}_{\mathbf{x}}\mathbf{F}(x,p)|_{\mathbf{x}=\mathbf{y}_n, p=p_0}, \quad \mathbf{K} = \mathbf{D}_p\mathbf{F}(\mathbf{x},p)|_{\mathbf{x}=\mathbf{y}_n, p=p_0} . \quad (4)$$

Without control, i.e., $\Delta p_n = 0$, the trajectory $\mathbf{x}_i$ ($i = n+1, \ldots$) diverges from the target orbit $\mathbf{y}_i$ ($i = n+1, \ldots$) exponentially. The task is to program the parameter perturbations Δp_n so that $|\mathbf{x}_i - \mathbf{y}_i| \to 0$ for subsequent iterates $i \geq n+1$.

For each point on the target orbit, there exist both a stable and an unstable direction [20]. These directions can be calculated by using the numerical method of Ref. [20]. The calculated stable and unstable directions are stored together with the target orbit, all of which are to be used to compute the parameter perturbations applied at each time step. Let $\mathbf{e}_{s(n)}$ and $\mathbf{e}_{u(n)}$ be the stable and unstable directions at $\mathbf{y}_n$ and, $\mathbf{f}_{s(n)}$ and $\mathbf{f}_{u(n)}$ be two vectors perpendicular to $\mathbf{e}_{u(n)}$ and $\mathbf{e}_{s(n)}$, respectively. The vectors $\mathbf{f}_{s(n)}$ and $\mathbf{f}_{u(n)}$ are also called the contravariant vectors [11], which satisfy $\mathbf{f}_{u(n)}\cdot\mathbf{e}_{u(n)} = \mathbf{f}_{s(n)}\cdot\mathbf{e}_{s(n)} = 1$ and $\mathbf{f}_{u(n)}\cdot\mathbf{e}_{s(n)} = \mathbf{f}_{s(n)}\cdot\mathbf{e}_{u(n)} = 0$. To stabilize $\{\mathbf{x}_n\}$ around $\{\mathbf{y}_n\}$, we require the next iteration of $\mathbf{x}_n$, after falling into a small neighborhood around $\mathbf{y}_n$, to lie on the stable direction at $\mathbf{y}_{(n+1)}(p_0)$, i.e.,

$$[\mathbf{x}_{n+1} - \mathbf{y}_{(n+1)}(p_0)]\cdot\mathbf{f}_{u(n+1)} = 0 . \quad (5)$$

Substituting Eq. (3) into Eq. (5), we obtain the following expression for the parameter perturbation:

$$\Delta p_n = \frac{\{\mathbf{J}\cdot[\mathbf{x}_n - \mathbf{y}_n(p_0)]\}\cdot\mathbf{f}_{u(n+1)}}{-\mathbf{K}\cdot\mathbf{f}_{u(n+1)}} , \quad (6)$$

where if $\Delta p_n > \delta$, we set $\Delta p_n = 0$.

In stabilizing unstable periodic orbits, the average transient ("waiting") time to achieve the control scales with the maximum allowed parameter perturbation δ as $\tau \sim \delta^{-\gamma}$, where the scaling exponent γ can be computed in terms of the stable and unstable eigenvalues of the unstable periodic orbits [11]. For cases where $\gamma > 1$ (typical for two-dimensional maps [11]), the transient time can be significantly reduced if somewhat larger parameter perturbations are allowed. The problem of transient time is *much less severe* in our case, since our target orbit is very long. In principle, when the trajectory enters the neighborhood of any one of the points on the target orbit, parameter control Eq. (6) can be applied. Thus, even if the size of every neighborhood around the target orbit is small, the transient time required can be significantly reduced by increasing the length of the target orbit.

III. NUMERICAL RESULTS

To illustrate control, we use the following Ikeda-Hammel-Jones-Maloney map [4],

$$z_{n+1} = A + Bz_n \exp\left[ik - \frac{ip}{1+|z_n|^2}\right], \tag{7}$$

which is an idealized model of an optical ring cavity [4], where $z = x + iy$ is a complex number. Equation (7) thus defines a two-dimensional map $(x_n, y_n) \rightarrow (x_{n+1}, y_{n+1})$. We choose p to be the control parameter. For parameter values $A=0.85$, $B=0.9$, $k=0.4$, and $p_0=7.3688$, there is a chaotic attractor, as shown in Fig. 2. The maximum Lyapunov exponent of the chaotic attractor is $\lambda_{\max} \approx 0.431$ and the size of the whole attractor is $L \sim 4.0$. Thus the time required for perturbations of magnitude $\epsilon = 10^{-5}$ to grow to the size of the attractor is $T \sim \ln(L/\epsilon)/\lambda_{\max} \approx 30$ iterations. On the attractor, there are clearly two distinct components. One is the part roughly in the center of the plot with higher probability density, i.e., orbits tend to visit this part of the attractor more frequently. The other part is the outer part with lower probability density. The coexistence of these two distinct parts of the attractor in the phase space gives rise to the intermittently chaotic time series shown in Fig. 1. Dynamically, there is an interior crisis at $p_c \approx 7.2688$ [1,2]. For $p < p_c$, there is a small chaotic attractor that corresponds to the inner part of the attractor at $p=7.3688$, and also a nonattracting chaotic saddle that roughly corresponds to the outer part of the attractor at $p=7.3688$. At $p = p_c$, the small chaotic attractor collides with the chaotic saddle, resulting in a larger attractor which is approximately the union of the small chaotic attractor and the chaotic saddle. For $p > p_c$, trajectories visit both parts of the attractor, leading to intermittently chaotic time series.

Our goal is to stabilize trajectories on the inner part of the chaotic attractor. The first step is to construct a target orbit that wanders only on the inner part of the attractor. As we described in Sec. II, when the trajectory starting from some point falls on the outer part of the attractor within the observation time Δt, perturbations of the type $10^{-5}\sigma$, where σ is a uniform random variable in [0,1], are applied to that point. To guarantee that the small perturbation grows to the size of the attractor within Δt, we choose $\Delta t = 50 > T \approx 30$. Figure 3(a) shows 20 000 points on a target chaotic orbit on the inner attractor. Compared with the inner part of the original attractor (Fig. 2), there are gaps that appear to exist in all scales on the target attractor, as shown in a blowup of the target orbit in part of the phase space, in Fig. 3(b). These

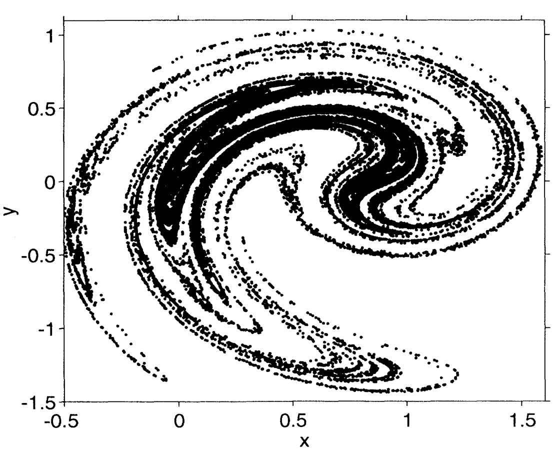

FIG. 2. A chaotic attractor of the Ikeda-Hammel-Jones-Moloney map at $A=0.85$, $B=0.9$, $k=0.4$, and $p=7.3688$. There are two distinct parts of the chaotic attractor. One is located approximately at the center of the plot with higher frequency of visits by the trajectory. This part corresponds to a chaotic attractor that exists before the interior crisis at $p = p_c \approx 7.2688$. The other distinct part is located exterior to the inner part with a much lower probability density of orbit visits. This part evolves from a nonattracting chaotic saddle that exists before the crisis.

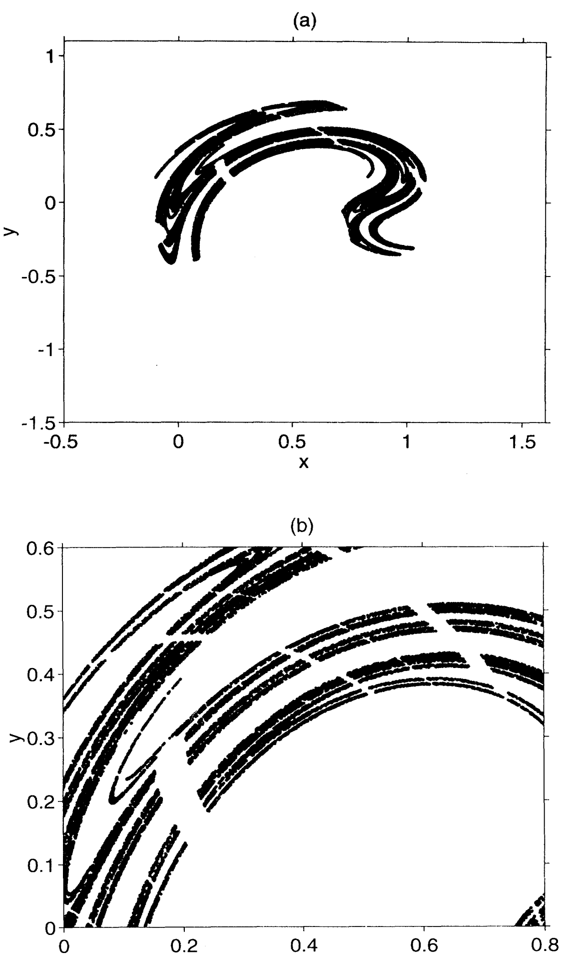

FIG. 3. (a) A long trajectory of the reconstructed target chaotic attractor that corresponds to the inner part of the attractor in Fig. 3. The observation time interval and the phase-space perturbation used to construct the target attractor are $\Delta t=50$ and $\epsilon = 10^{-5}$, respectively. (b) Blowup of the target orbit in part of the phase space.

gaps arise due to the complicated heteroclinic crossing of stable and unstable manifolds associated with the chaotic attractor and the chaotic saddle that exist before the crisis [2] and, therefore, these gaps are the "invaded" regions of the outer part of the attractor, which corresponds to the nonattracting chaotic saddle before the crisis.

To achieve the control, it is necessary to compute the stable and unstable directions ($\mathbf{e}_s$ and $\mathbf{e}_u$) for each point on the target orbit [20]. Figures 4(a) and 4(b) show the stable and unstable directions, respectively, for 5000 points on the target orbit. Knowing these directions, the stable and unstable contravariant vectors are computed straightforwardly by using $\mathbf{f}_{u(n)}\cdot\mathbf{e}_{u(n)}=\mathbf{f}_{s(n)}\cdot\mathbf{e}_{s(n)}=1$ and $\mathbf{f}_{u(n)}\cdot\mathbf{e}_{s(n)}=\mathbf{f}_{s(n)}\cdot\mathbf{e}_{u(n)}=0$. Parameter perturbations can then be computed from Eq. (6) at each time step.

Figure 5(a) shows a controlled trajectory (20 000 points) in the phase space starting from an arbitrarily initial condition. Initially, the trajectory is not close to the target orbit. As soon as it is within 10^{-3} of some point on the target orbit, parameter perturbations are applied to stabilize the trajectory around the target orbit. The time required for this to occur is usually a few hundred iterations (quite short indeed). The maximum allowed parameter perturbation is set to be $\delta=10^{-2}$. Visually, the controlled trajectory looks the same as the target orbit. Figures 5(b) and 5(c) show the controlled time series x_n and y_n, respectively, which are apparently within the desired part of the chaotic attractor. The undesired (outer) part of the original attractor is never visited by the controlled trajectory. Figure 5(d) shows the required parameter perturbation Δp_n as a function of the discrete time n. In general, the applied parameter perturbations are around $10^{-7.5}$, very small indeed. Occasionally, somewhat larger parameter perturbations are required (around 10^{-3}), which occurs when the trajectory gets very close to points on the target orbit where stable and unstable directions are almost identical (the so called tangency points) [21]. In principle, the target orbit can be made arbitrarily long, so the controlled desired trajectory can be arbitrarily long, accordingly. An alternative approach could be to construct a long recurrent orbit on the desired part of the chaotic attractor such that in any practical time scales the orbit is chaotic. Trajectories stabilized around the recurrent orbit would then exhibit desired chaotic behavior in practically relevant time scales.

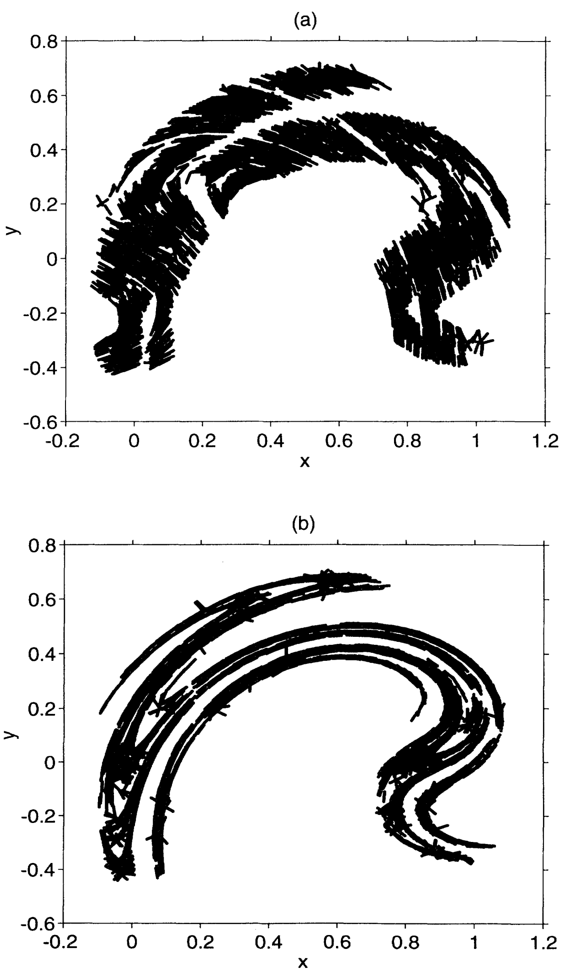

FIG. 4. Stable (a) and unstable (b) directions at 5000 points on the reconstructed target attractor.

IV. DISCUSSIONS

In this work, we have devised a scheme to stabilize trajectories around some distinct part of a chaotic attractor by applying small perturbations to a system parameter. The controlled trajectory would supposedly correspond to a better operational chaotic state of the system. Our feedback control method is based on the OGY idea of stabilizing unstable periodic orbits. The features of our work are twofold. First, we use the butterfly effect of chaotic systems to construct a target orbit on the desired part of the chaotic attractor. Second, we make use of the geometric structure, stable and unstable directions, along the target orbit to achieve the control. Our strategy thus allows us to select a desired state of the system to be stabilized, thereby avoiding the other, undesired part of the attractor, which corresponds to the undesired state of the system.

The method described is robust in the presence of small external noise. This is so because the target orbit is constructed by applying small random perturbations in the state variables when necessary and, consequently, the target orbit is actually a noisy chaotic orbit. The readiness with which the control is achieved, and the extremely small parameter perturbations required [Fig. 5(d)], suggest that our method works even in a noisy environment.

An appealing feature of our method lies in its potential relevance to biological systems. It is commonly believed now that some healthy states of biological systems may be chaotic. A known example is the human heart rate variability [7], where a healthy heart would generate very irregular, or chaotic, heart-rate variations in time. Regular or nonchaotic heart rate variations are usually generated in hearts with serious malfunctions. Thus, it is desirable to maintain chaotic heart-rate variations for cardiac patients. Nonetheless, despite the positivity of

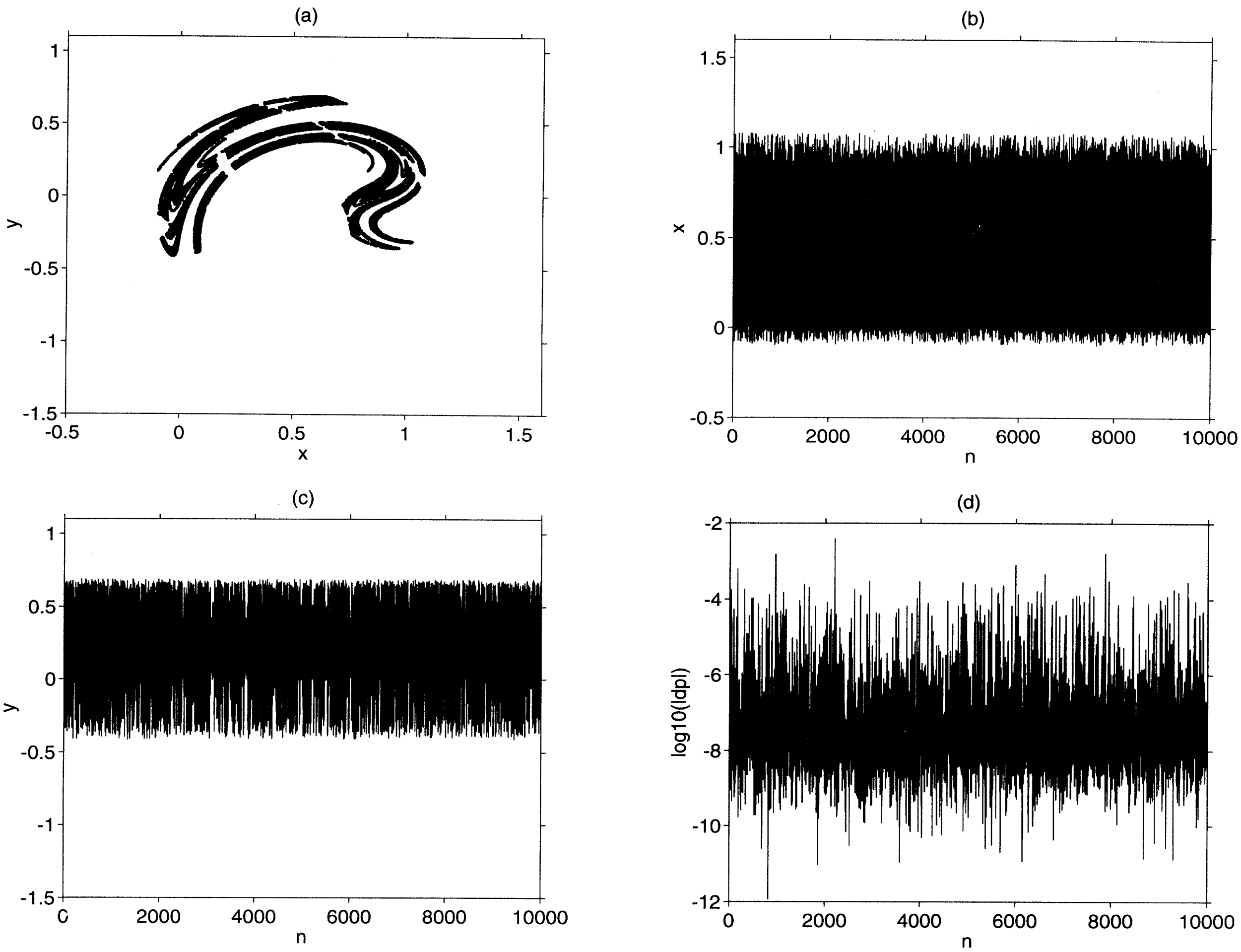

FIG. 5. (a) The controlled trajectory in the neighborhood of the target chaotic attractor. The trajectory starts from an arbitrary initial condition. When the trajectory is within 10^{-3} of some point on the target attractor, small parameter perturbations with magnitude less than 10^{-2} are applied to stabilize the trajectory around the target orbit. (b) and (c) Controlled time series x_n and y_n, respectively. It is clear that the controlled time series only contain the desired chaotic phase. The controlled trajectory never visits the outer attractor, which corresponds to the undesired chaotic phase. (d) The required parameter perturbation Δp_n, which is about $10^{-7.5}$ on the average. Occasionally, somewhat larger parameter perturbations (about 10^{-3}) are required, which occurs when the controlled trajectory is close to points where the stable and unstable directions are very close (tangency points).

chaos, some chaotic states may correspond to undesirable states of the system, as in certain types of epileptic seizures [5]. Therefore, in this case it is desirable to keep the system running in the favored chaotic state to avoid the unfavored chaotic state. The present work has demonstrated that this is possible, at least for very simple chaotic systems. While our numerical examples are performed using a well studied low-dimensional chaotic system, we believe that our idea may have potential relevance to more complicated biological and physical systems where certain chaotic states are desired.

Finally, we remark that the algorithm presented in this paper applies well when a system's equations are known. In experiments it is usually the case that only a measured time series is available. It is then necessary to use the delay-coordinate embedding technique [22] to extract quantities required to compute the parameter perturbations, such as the stable and unstable directions along the target orbit. While calculating such quantities for low-periodic orbits embedded in a chaotic attractor is relatively easy [11,13], it is not clear at present that this can be easily done for a long target orbit embedded in a chaotic attractor. Therefore, there is currently no assurance that our technique can be readily applied to real experimental systems where the equations are not available. Nonetheless, we hope that the method in this paper will stimulate work in the stabilizing of chaotic states in experiments.

ACKNOWLEDGMENTS

This work was supported by the Kansas Institute for Theoretical and Computational Science through the K*STAR/NSF program. We thank David Lerner and Mark Frei for valuable discussions regarding the epileptic seizures.

[1] C. Grebogi, E. Ott, and J. A. Yorke, Physica **7D**, 181 (1983); C. Grebogi, E. Ott, F. Romeiras, and J.A. Yorke, Phys. Rev. A **36**, 5365 (1987).

[2] Y. C. Lai, C. Grebogi, and J. A. Yorke, in *Applied Chaos,* edited by J. H. Kim and J. Stringer (Wiley, New York, 1992), pp. 441–455.

[3] K. G. Szabó and T. Tél, Phys. Rev. E **50**, 1070 (1994).

[4] S. M. Hammel, C. K. R. T. Jones, and J. V. Moloney, J. Opt. Soc. Am. B **2**, 552 (1985).

[5] L. D. Iasemidis, J. C. Sackellares, H. P. Zaveri, and W.J. Williams, Brain Topogr. **2**, 187 (1990); L. D. Iasemidis and J. C. Sackellares, in *Measuring Chaos in the Human Brain,* edited by D. W. Duke and W. S. Pritchand (World Scientific, Singapore, 1991), pp. 49–82.

[6] J. Z. Tepper and A. J. Mandell, Ann. N.Y. Acad. Sci. **504**, 88 (1987); T. Chang, S. J. Schiff, T. Sauer, J.-P. Gossard, and R. E. Burke, Biophys. J. **67**, 671 (1994); S. J. Schiff, K. Jerger, T. Chang, T. Sauer, and P. G. Aitken, *ibid.* **67**, 684 (1994).

[7] A known example where a chaotic state may actually correspond to healthy physiological condition is the human heart-rate variability. Goldberger, Mandell, West, and their collaborators discovered that healthy hearts in fact generate highly irregular behavior in the plot of instantaneous heart rate versus time. In contrast, patients with serious heart diseases usually exhibit relatively regular heart-rate variability. They have performed numerous clinical studies to verify their theory. The implication of their discovery is that healthy hearts possess much more complexity (chaos) than sick hearts. See the following papers: A. L. Goldberger, L. J. Findley, M. R. Blackburn, and A. J. Mandell, Am. Heart J. **107**, 612 (1984); A. L. Goldberger, V. Bhargava, B. J. West, and A. J. Mandell, Physica D **19**, 282 (1986); A. J. Mandell, *ibid.* **27**, 235 (1987); A. L. Goldberger and B. J. West, Ann. N.Y. Acad. Sci. **504**, 195 (1987); A. L. Goldberger, D. R. Rigney, J. Mietus, E. M. Antman, and S. Greenwald, Experientia **44**, 983 (1989); C. K. Peng, J. Mietus, J. M. Hausdorff, S. Havlin, H. E. Stanley, and A. L. Goldberger, Phys. Rev. Lett. **70**, 1343 (1992).

[8] Recently, Schiff *et al.* have performed experiments to control chaos in rat brain slices [S. J. Schiff, K. Jerger, D. H. Duong, T. Chang, M. L. Spano, and W. L. Ditto, Nature **370**, 615 (1994)]. They demonstrated that the chaotic bursting behavior during periods of time in between epileptic seizures can be made less chaotic or even periodic by applying small electrical pulses to the brain slices. The design of the control is based on the OGY idea of controlling chaos. They also demonstrated that the bursting behavior can be made less periodic by applying anticontrol.

[9] C. Skarda and W. J. Freeman, Behav. Brain Sci. **10**, 161 (1987); G. Mayer-Kress and S. P. Layne, Ann. N.Y. Acad. Sci. **62**, (1987); P. E. Rapp, T. R. Bashove, J. M. Martineria, A. M. Albano, and A. I. Mees, Brain Topogr. **2**, 99 (1989); T. M. McKenna, T. A. McMullen, and M. F. Shlesinger, Neuroscience **60**, 587 (1994).

[10] D. Lerner, M. Frei, J. Ralston, J. Dorfmeister, and I. Osorio (private communication).

[11] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990); in *Chaos: Soviet-American Perspectives on Nonlinear Science,* edited by D. K. Campbell (AIP, New York, 1990).

[12] For example, F. J. Romeiras, C. Grebogi, E. Ott, and W. Dayawansa, Physica **58D**, 165 (1992); U. Dressler and G. Nitsche, Phys. Rev. Lett. **68**, 1 (1992); D. Auerbach, C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. Lett. **69**, 3479 (1992); Y. C. Lai, M. Ding, and C. Grebogi, Phys. Rev. E **47**, 86 (1993); Y. C. Lai and C. Grebogi, *ibid.* **49**, 1094 (1994); W. Yang, M. Ding, A. J. Mandell and E. Ott, *ibid.* **51**, 102 (1995).

[13] W. L. Ditto, S. N. Rauseo, and M. L. Spano, Phys. Rev. Lett. **65**, 3211 (1990); J. Singer, Y.-Z. Wang, and H.H. Bau, *ibid.* **66**, 1123 (1991); A. Azevedo and S. M. Rezende, *ibid.* **66**, 1342 (1991); E. R. Hunt, *ibid.* **67**, 1953 (1992); R. Roy, T. W. Murphy, Jr., T. D. Maier, and Z. Gills, *ibid.* **68**, 1259 (1992); C. Reyl, L. Flepp, R. Badii, and E. Burn, Phys. Rev. E **47**, 267 (1993); T. C. Newell, P. M. Alsing, A. Gavrielides, and V. Kovanis, Phys. Rev. Lett. **72**, 1647 (1994); Phys. Rev. E **49**, 313 (1994).

[14] B. Peng, V. Petrov, and K. Showalter, J. Chem. Phys. **95**, 4957 (1991); V. Petrov, V. Gáspár, J. Masere, and K. Showalter, Nature **361**, 240 (1993); V. Petrov, M. J. Crowley, and K. Showalter, Phys. Rev. Lett. **72**, 2955 (1994).

[15] A. Garfinkel, W. L. Ditto, M. L. Spano, and J. Weiss, Science **257**, 1230 (1992).

[16] Y. C. Lai and C. Grebogi, Phys. Rev. E **47**, 2357 (1993).

[17] N.J. Mehta and R. M. Henderson, Phys. Rev. A **44**, 4861 (1991).

[18] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, 1993).

[19] T. Shinbrot, E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **65**, 3215 (1990); E. J. Kostelich, C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. E **47**, 305 (1993).

[20] Y. C. Lai, C. Grebogi, J. A. Yorke, and I. Kan, Nonlinearity **6**, 779 (1993).

[21] Hammel demonstrated in his algorithm of reducing noise for chaotic attractors that the effect of noise may become very severe for trajectory points very close to the tangency points of stable and unstable manifolds [S. M. Hammel, Phys. Lett. A **148**, 421 (1990)].

[22] N. Packard, J. Crutchfield, J. D. Farmer, and R. Shaw, Phys. Rev. Lett. **45**, 712 (1980); F. Takens, *Detecting Strange Attractors in Turbulence,* Lecture Notes in Mathematics Vol. 898 (Springer-Verlag, New York, 1981), p. 366.